JOURNAL OF COMPUTATIONAL PHYSICS 14, 301-317 (1974)

Analytic Computation of Some Integrals in Fourth Order Quantum Electrodynamics*

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Received June 18, 1973

A program for the analytic evaluation of some parametric integrals which occur in fourth order QED calculations is described.

1. INTRODUCTION

It has been recognized for some time that the precise computations of the fundamental quantities which appear in quantum electrodynamics such as the Lamb shift and anomalous magnetic moment require far more accurate evaluation of the integrals which arise in their calculation than is possible with current numerical techniques. Clearly the ideal way of computing these integrals is to evaluate them exactly by analytic means. In this paper we consider one approach to this problem in which such integrals are evaluated exactly by computer. Until quite recently, this was not believed possible for integrals with the complexity which one finds in such calculations except by heuristic means which do not always guarantee a solution to the problem [1, 2]. However, in 1968 it was shown by Risch [3] that a procedure existed which would provide for the algorithmic evaluation of integrals of expressions involving a much wider class of functions than had been considered by such means previously. In particular, Moses [4] showed that a decision procedure existed for the indefinite integration of expressions involving Spence functions. This method will apply as a special case to the evaluation of integrals of expressions of the form

$$f(x) = \sum_{n} R(x)\, L_n(x), \tag{1}$$

* Work supported in part by the National Science Foundation under Grant No. GJ-32181. Computer time supported by the Advanced Research Projects Agency of the Office of the Department of Defense under Contract No. F30602-70-C0300 at the University of Utah.

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where $R(x)$ is a rational function of x and

$$L_n(x) = \int_1^x L_{n-1}(t)/(t-1)\, dt, \qquad (2)$$
$$L_0(t) = (t-1)/t$$

defines the Spence functions. It then determines the integral of such expressions analytically provided that the integral is still an expression of the same form as Eq. (1). To integrate such expressions we begin by expanding $R(x)$ into a sum of partial fractions. If the partial fraction denominators are linear, the resulting sum can then be integrated term by term using Eq. (2). However, if denominators of higher order occur, then the resulting integrals will no longer be Spence functions with simple rational arguments and therefore not transformable into the same form as Eq. (1). Further integration of such functions would then be extremely difficult in general. Such results can be applied immediately to the consideration of all relevant definite integrals which arise in computing some fourth order vertex diagrams in quantum electrodynamics, which were shown by Karplus and Kroll [5] to be classifiable in terms of five indices. Specifically, such integrals take the form [6]:

$$I(k, m, n, r, s) = \int_0^1 dy \int_0^1 dx \int_0^1 dw \int_0^1 dz \, \frac{z^r w^s x^{2k+n-m-3} y^{k+n-2} \Delta^m}{\beta^n \Gamma^k} \qquad (3)$$

where

$$\begin{aligned} \beta &= 1 - xy(1 - wz(1 - wz)), \\ \Delta &= 1 - x(1 - wz(1 - w)), \\ \Gamma &= w^2 x\beta + y\Delta^2. \end{aligned} \qquad (4)$$

The important point about these functions is that it is possible to carry through the integrations in such a way that the denominators which result after any given integration can always be expressed in terms of linear factors with respect to the next variable of integration, and hence by the above method we can arrive at a final result in terms of a few particular Spence functions.

The results of each integration are found by appropriate substitutions of the end point values into the indefinite forms which remain after the partial fraction expansions. A further difficulty can now arise because of singularities introduced in the definite integrals by end point divergences. These are handled in the present case by a standard regularization procedure as discussed in Section 4.

In this paper we describe a program which has been developed in REDUCE [6] for the analytic evaluation of all such integrals. This is not however the first such program for doing this; a pioneering effort in this direction was that of Peterman [7] who developed a program written in SCHOONSCHIP [8] for evaluating this class of integrals. Our particular program in fact draws heavily on

Peterman's methods. However, his published program uses previously computed tables of integrals for the evaluation of many of the forms encountered. This in fact limits the use of the program to those particular parameter values accommodated by the tables. Peterman's table is however complete enough to consider all integrals one would normally meet in physical calculations. Our approach, on the other hand, involves the use of algorithmic procedures for single dimensional integral evaluation which apply to a much wider range of parameters than considered by Peterman's tables and uses only a few tabulated constants for its evaluation. We have followed this alternative approach because we believe that the extension of these ideas to higher orders of quantum electrodynamics or to other fields will be more easily accomplished by this approach than by table lookup. We shall say more about the problem of higher order integral evaluation later.

The outline of this paper is as follows. In Section 2 we show why the integrals which occur in some fourth-order vertex graphs may be represented in the form of Eq. (3). Next, the actual integration procedure is discussed in terms of the generalized polylogarithms of Nielsen [7, 9, 10] which turn out to be more convenient for our purposes than the Spence functions. In Section 4, we discuss the form of the program which we have developed, and we conclude finally with some remarks on the extension of these ideas to higher orders of QED. The actual program for doing the integrals is given in Appendices A and B, with various examples.

2. Form of the Integrals

Expressions of the form $I(k, m, n, r, s)$ are found to contribute to many fourth order processes in QED. In particular, the fourth order ladder contributions to the anomalous magnetic moment of the electron may be expressed completely as a sum of such terms. To illustrate this, we shall consider the evaluation of the crossed ladder vertex diagram in Fig. 1. We have chosen this example because it is irreducible and therefore the hardest to compute by standard methods. The unrenormalized amplitude corresponding to this diagram is, according to the usual Feynman rules:

$$\begin{aligned}
&-ie\bar{u}\left(p+\frac{q}{2}\right) J_\mu^{\text{cross}} u\left(p-\frac{q}{2}\right) \\
&\quad = ie \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4}\, \bar{u}\left(p+\frac{q}{2}\right) \gamma_\sigma(\not{p}_4+m)\,\gamma_\rho(\not{p}_3+m)\,\gamma_\mu(\not{p}_2+m) \\
&\qquad \times \gamma^\sigma(\not{p}_1+m)\,\gamma^\rho u\left(p-\frac{q}{2}\right) \frac{1}{k_1^2-\lambda^2}\,\frac{1}{k_2^2-\lambda^2} \prod_{i=1}^{4} \frac{1}{p_i^2-m^2},
\end{aligned} \tag{5}$$

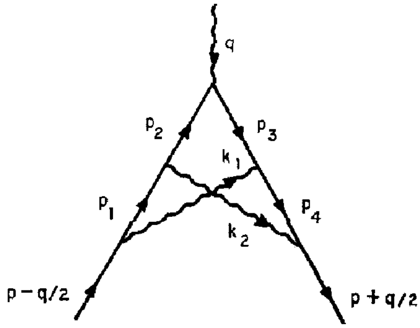

FIG. 1. Fourth order crossed ladder vertex graph in QED.

where

$$
\begin{aligned}
p_1 &= p - \frac{q}{2} - k_1 , \\
p_2 &= p - \frac{q}{2} - k_1 - k_2 , \\
p_3 &= p + \frac{q}{2} - k_1 - k_2 , \\
p_4 &= p + \frac{q}{2} - k_2 .
\end{aligned} \tag{6}
$$

One way to evaluate the integral in Eq. (5) relies on combining its six denominators into parametric form using

$$
\frac{1}{abcdef} = \frac{5! \int_0^1 du \int_0^1 z\, dz \int_0^1 w^2\, dw \int_0^1 x^3\, dx \int_0^1 y^4\, dy}{[(a-b)\, uzwxy + (b-c)\, zwxy + (c-d)\, wxy + (d-e)\, xy + (e-f)\, y + f]^6}, \tag{7}
$$

where

$$
\begin{aligned}
a &= p_3^2 - m^2, \\
b &= p_2^2 - m^2, \\
c &= p_1^2 - m^2, \\
d &= k_1^2 - \lambda^2, \\
e &= p_4^2 - m^2, \\
f &= k_2^2 - \lambda^2.
\end{aligned} \tag{8}
$$

This formula is a special case of the more general relation

$$\frac{1}{a_1 \cdots a_n}$$
$$= \frac{(n-1)! \int_0^1 dx_1 \int_0^1 x_2\, dx_2 \cdots \int_0^1 x_{n-1}^{n-2}\, dx_{n-1}}{[(a_1 - a_2) \prod_{i=1}^{n-1} x_i + (a_2 - a_3) \prod_{i=1}^{n-2} x_i + \cdots + (a_{n-1} - a_n)\, x_{n-1} + a_n]^n}. \tag{9}$$

Before the parametric integrals can be evaluated, the integrations over the internal photon momenta must be carried out. To facilitate this, the denominator of Eq. (7) must be diagonalized with respect to k_1 and k_2; in other words it must be brought into quadratic form in these variables so that no terms of the form $k_1 \cdot k_2$, $k_1 \cdot p$, $k_2 \cdot p$, etc. remain. The transformation which effects this diagonalization is

$$\begin{aligned} k_1 &= k_1' - wzk_2' + c_1 q + c_2 p, \\ k_2 &= k_2' + c_3 q + c_4 p, \end{aligned} \tag{10}$$

where

$$\begin{aligned} c_1 &= -w/2 + uwz - wzc_3, \\ c_2 &= w - wyz\Delta/\beta, \\ c_3 &= \tfrac{1}{2} y\Delta/\beta - c_5, \\ c_4 &= y\Delta/\beta, \\ c_5 &= wxyz(1 - w - u(1 - wz))/\beta. \end{aligned} \tag{11}$$

The denominator in Eq. (7) then becomes

$$(xyk_1'^2 + \beta k_2'^2 - y\Gamma m^2/\beta + c_6\lambda^2 + c_7 q^2)^6, \tag{12}$$

where

$$\begin{aligned} c_6 &= wxy - xy + y - 1, \\ c_7 &= uw^2xyz(1 - uz) + y(1 - x + uwxz(1 - wz))\, c_5. \end{aligned} \tag{13}$$

The integrations over the photon momenta are now straightforward.

For our present purposes, we limit ourselves to integrals in which $\lambda \to 0$ and $q^2 = 0$. Letting the photon mass λ approach 0 restricts us to considering only infrared convergent integrals. By investigating integrals in which $q^2 = 0$ we can evaluate the Lamb shift and the anomalous magnetic moment since both are proportional to quantities evaluated at zero momentum transfer. When both λ^2 and q^2 are removed from Eq. (12), the integrals simplify greatly and consequently the integration over the variable u is straightforward. All contributions to the matrix elements in Eq. (5) are then found to be a sum of integrals of the form:

$$I(k, m, n, r, s) = \int_0^1 w^s \, dw \int_0^1 z^r \, dz \int_0^1 x^{2k+n-m-3} \Delta^m \, dx \times \int_0^1 \frac{y^{k+n-2}}{(1 - x_3 y)^n (x_1 + x_2 y)^k} \, dy, \tag{14}$$

where

$$\begin{aligned} x_1 &= w^2 x, \\ x_2 &= \Delta^2 - w^2 x^2 (1 - wz(1 - wz)), \\ x_3 &= x(1 - wz(1 - wz)) \end{aligned} \tag{15}$$

in agreement with Eq. (3).

In Eq. (14) the original denominator has been factored into two terms each of which is linear in y. One could now use standard partial fraction techniques to carry out the x and y integrations. A more efficient method results for our calculation, however, by the observation that all denominator combinations possible for Eq. (14) when n and k are both greater than zero can be obtained by suitable differentiation with respect to B and C of the following trivial integral:

$$\begin{aligned} &\int_0^1 \frac{y^{k+n-2} \, dy}{(1 - Cy)^n (A + By)^k} \\ &= \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial^{n-1}}{\partial C^{n-1}} \frac{\partial^{k-1}}{\partial B^{k-1}} \int_0^1 \frac{dy}{(1 - Cy)(A + By)} \\ &= \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial^{n-1}}{\partial C^{n-1}} \frac{\partial^{k-1}}{\partial B^{k-1}} \frac{1}{(AC + B)} \ln \frac{A + B}{A(1 - C)} . \end{aligned} \tag{16}$$

Further, when the expressions for $A + B$ and $A(1 - C)$ are evaluated they factor into pieces which are linear in x. The x integration can then also be carried out with the help of a generating function and yields combinations of rational functions and logarithms in the variables w and z. These functions depend on the parameters k, m, n and thus we can write

$$I(k, m, n, r, s) = \int_0^1 z^r \, dz \int_0^1 w^s \, dw \, SI(k, m, n), \tag{17}$$

where

$$SI(k, m, n) = \int_0^1 x^{2k+n-m-3} \Delta^m \, dx \int_0^1 \frac{y^{k+n-2}}{(1 - x_3 y)^n (x_1 + x_2 y)^k} \tag{18}$$

is no longer a function of x and y.

3. Integration over X and Y Using Subtracted Nielsen Functions

The evaluation of the various $SI(k, m, n)$ can be performed most efficiently within the framework of the so-called subtracted Nielsen functions [7, 9, 10]. Ordinary Nielsen functions are defined by their expansion for real x and $|x| < 1$:

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{\ln^{n-1} t \ln^p(1-xt)}{t}\,dt$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i\, St_{p+i}^{(p)}}{(p+i)!} \frac{x^{j+p}}{(p+i)^n}, \tag{19}$$

where the coefficients $St_i^{(P)}$ are the well known Stirling numbers of the first kind, defined by the factorial polynomials

$$k^{(i)} \equiv \frac{k!}{(k-i)!} = \sum_{p=0}^{n} St_i^{(p)} k^p \tag{20}$$

or by

$$\ln^p(1+x) = p! \sum_{i=p}^{\infty} St_i^{(p)} \frac{x^i}{i!}. \tag{21}$$

The mth subtracted Nielsen function is defined by

$$S_{n,p}^{m,m}(x) = \sum_{i=0}^{\infty} \frac{(-1)^{m-p+i}\, St_{m+i}^{(p)} x^i}{(m+i)!\,(m+i)^n}, \tag{22}$$

and is formed from $S_{n,p}(x)$ by subtracting off the first $m - p$ terms and dividing the result by x^m, viz.

$$S_{n,p}^{m,m}(x) \equiv \frac{1}{x^m} \left(\sum_{i=0}^{\infty} \frac{(-1)^i\, St_{p+i}^{(p)} x^{p+i}}{(p+i)!\,(p+i)^n} - \psi_{n,p}^m(x) \right) \tag{23}$$

where

$$\psi_{n,p}^m(x) = \sum_{i=0}^{m-p-1} \frac{(-1)^i\, St_{p+i}^{(p)} x^{p+i}}{(p+i)!\,(p+i)^n}. \tag{24}$$

The first term surviving in this subtraction procedure is

$$\frac{1}{x^m} \left[\frac{(-1)^{m-p}\, St_m^{(p)} x^m}{m!\, m^n} \right]$$

so that Eq. (23) can be written in the form of Eq. (22). Finally, the generalized subtracted Nielsen function is defined by

$$S_{n,p}^{m,k}(x) = \sum_{i=0}^{\infty} \frac{(-1)^{m-p+i}\, St_{m+i}^{(p)} x^i}{(m+i)!\,(k+i)^n}. \tag{25}$$

These functions are particularly useful for two reasons. First, as will be seen shortly, they provide a compact representation for the $SI(k, m, n)$ and secondly they possess straightforward integration properties.

Of particular interest in the present calculation are the forms $S_{0,1}^{n,n}$, or subtracted logarithms, defined by

$$\begin{aligned} -Sl_n(x) = S_{0,1}^{n,n} &= \sum_{i=0}^{\infty} \frac{(-1)^{n-1+i}\, St_{i+1}^{(1)} x^i}{(n+i)!} \\ &= \frac{1}{x^n}\left[\sum_{i=0}^{\infty}\left\{\frac{(-1)^i\, St_{i+1}^{(1)} x^{i+1}}{(i+1)!} - \sum_{i=0}^{m-2} \frac{(-1)^i\, St_{i+1}^{(1)} x^{i+1}}{(i+1)!}\right\}\right]. \end{aligned} \tag{26}$$

Recognizing that $St_{i+1}^{(1)} = (-1)^i\, i!$, this gives us

$$\begin{aligned} -Sl_n(x) &= \frac{1}{x^n}\left[\sum_{i=0}^{\infty} \frac{x^{i+1}}{i+1} - \sum_{i=0}^{n-2} \frac{x^{i+1}}{i+1}\right] \\ &= \frac{1}{x^n}\left[\sum_{i=1}^{\infty} \frac{x^i}{i} - \sum_{i=1}^{n-1} \frac{x^{i+1}}{i+1}\right], \end{aligned} \tag{27}$$

or

$$Sl_n(x) = \frac{1}{x^n}\,[\ln(1-x) + \psi_1^{(n)}(x)], \tag{28}$$

where

$$\psi_i^{(n)}(x) = \sum_{j=1}^{n-1} \frac{x^j}{j^i}. \tag{29}$$

Also of interest are the subtracted dilogarithms

$$Sd_n(x) = S_{1,1}^{n,n} = \frac{1}{x^n}\,[Li_2(x) - \psi_2{}^n(x)]. \tag{30}$$

The integration properties which make these subtracted polylogarithms so useful are

$$\int_0^1 dy\, y^{m-1} Sl_n(xy) = [Sl_n(x) - Sl_m(x)]/(n-m), \tag{31}$$

which follows readily from the definition Eq. (28) and

$$\int_0^1 dy\, y^{m-1} Sd_n(xy) = [Sd_n(x) - S_{1,1}^{n;m}(x)]/(n-m). \tag{32}$$

In general the second integral is not very useful except in particular cases. For example, when $m = 1$ and $x = 1$,

$$S_{1,1}^{n;1}(1) = \sum_{i=0}^{\infty} \frac{(-1)^{n+1+i} St_{n+i}^{(1)}}{(n+i)!\,(i+1)} = \sum_{i=0}^{\infty} \frac{1}{(n+i)(i+1)} = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{i}. \tag{33}$$

Armed with this basic information about the subtracted polylogarithms it is now possible to see how they provide a compact representation for the $SI(k, m, n)$. What one does is to expand all rational functions and rational coefficients of logarithms by partial fractions. One then expresses all such functions as far as possible in terms of subtracted logarithms, *eg*,

$$\frac{1}{w^n} \ln(1 - wz) = z^n Sl_n(wz) - w^{-n} \psi_1{}^n(wz). \tag{34}$$

When the w integration is completed, one is left with rational functions of z, subtracted logarithms in z and perhaps some dilogarithms. These functions are again expanded using partial fractions, and finally the integration over z is carried out, completing the required evaluation.

4. Structure of the Program

The program RSIN, which is given in Appendix A, is set up as follows. The first section contains definitions of frequently used algebraic procedures. Next are the rules necessary to do the x and y integration. Fortunately, as we mentioned earlier, all of the y and most of the x integrals can be done by suitably differentiating two generating functions. These functions are

$$\int_0^1 \frac{dv}{(1 - Ay)(1 - By)} = \frac{1}{A - B} (\ln(1 - B) - \ln(1 - A)) \tag{35}$$

and

$$\begin{aligned} \int_0^1 & \frac{x\, dx}{(1 - Dx)(1 - C_1x)(1 - C_2x)} \\ &= \frac{1}{(C_1 - C_2)} \frac{1}{(D - C_1)} (\ln(1 - C_1) - \ln(1 - D)) \\ &\quad - \frac{1}{(C_1 - C_2)} \frac{1}{(D - C_2)} (\ln(1 - C_2) - \ln(1 - D)). \end{aligned} \tag{36}$$

Next the rules needed to perform the integrals over w and z are given. These are more complicated in that rules for integrating elementary functions, logarithms, subtracted logarithms and dilogarithms are required.

An important feature of these integration rules is the regularization procedure we have adopted. All integrals handled by this program are globally finite, but because of the way they are split up, isolated singularities in the variables w and z can appear. These are of the form $\int_0^1 dw/w^n$, $\int_0^1 dw/(1-w)^n$, $\int_0^1 dz/(2z-1)^n$ and $\int_0^1 \ln z/z^n\,dz$. Such singularities are always compensated however by an equal and opposite term in another piece of the integral. For example, with a term like $C\int_0^1 dw/w^n$ there will always be a term $-C\int_0^1 dw/(1-w)^n$. Thus one has a choice to make; namely whether or not to retain these singular terms until the end of the calculation or throw them away as they arise. Like Peterman [6], we opt for the latter method in this paper. Thus the procedures for integrating such obviously divergent functions as $\int_0^1 Li_2(z)/z^n\,dz$ are set up to return the Hadamard finite part (HFP) of such integrals. The method of extracting the HFP is unique, as we shall illustrate by considering the above integral. Since

$$Li_2(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^2}, \tag{37}$$

this integral diverges for all $n < -1$. It is obvious that for finite n there are a finite number of singular terms in the integrand; in fact precisely $n-1$ of them as given by

$$\sum_{i=1}^{n-1} z^{i-n} = z^{1-n} + z^{2-n} + z^{3-n} + \cdots + z^{(n-1)-n}. \tag{38}$$

The Hadamard finite part is therefore found by subtracting these $n-1$ terms from the original integrand. Thus

$$HFP \int_0^1 \frac{Li_2(z)\,dz}{z^n} = \int_0^1 \frac{(Li_2(z) - \sum_{k=1}^{n-1} z^k/k^2)\,dz}{z^n} = \int_0^1 Sd_n(z)\,dz. \tag{39}$$

This example also illustrates the obvious connection between the HFP's and subtracted polylogarithms.

A program which computes the Hadamard finite parts is contained in Appendix B, with some examples. This program, called ZINT, is a subprogram of RSIN.

After the integration rules have been tabulated the procedure RSIN is defined. RSIN is a function of four variables FN, k, m and n. The variables are chosen to reflect the fact that all integrals have the structure

$$FN{*}SI(k, m, n)$$

where FN is polynomial in w and z and $SI(k, m, n)$ is as defined earlier in Eq. (18). The fundamental thrust of the procedure is to take the integrand specified by FN, k, m, n, expand it using elementary partial fraction decomposition rules and then recombine it as soon as possible into combinations of logarithms, rational functions and subtracted Nielsen functions, which are fed through the integration sequence.

In order to keep the integral table as small as possible, we make use of the fact that the integrals over w and z are invariant under the reflections $w \leftrightarrow 1 - w$ and $z \leftrightarrow 1 - z$ respectively. This optimization may be seen at various points in the definition of RSIN.

The integration tables provided are sufficient for the complete evaluation of any integral of the form given in Eq. (3). However, in order to allow for the use of the tables in the computation of integrals outside the range of the parameters defined by Eq. (3), (for example, a more general exponent for x in the integrand numerator) the table provides answers in these cases in terms of undefined integrals INT1, INT2 and INT3. The explicit forms for these integrals would have to be added to the table if integration of such functions was required, but this is straightforward.

Because of the dependence of the program on pattern matching for performing many of the operations involved, especially partial fraction decompositions, the program is not as efficient as one written specifically for such calculations. However, the times taken for the examples in Appendix A are not unreasonable considering that only a modest number of such integrations would be performed in any given physical calculation. The actual times in seconds for these examples for execution on the USC-ISI PDP-10 using 70K words of 1μ sec core are given in Table I. These times should be improved dramatically when a partial fraction package is made available in REDUCE in the near future.

TABLE I

Times in Sec. for Computing Various $I(k, m, n, r, s)$ on USC-ISI PDP-10 in 70K Words

k	m	n	r	s	Time
1	0	1	1	2	18.5
1	1	2	1	3	51.2
1	1	3	1	3	82.2
1	1	3	1	2	73.1
1	0	2	2	3	49.2

5. Conclusion

The methods which we have considered in this paper for analytic integration could clearly be applied to a wider range of problems than are considered here. Moreover, the Risch method as expounded by Moses [4] would provide for the evaluation of integrals in which the Spence functions were themselves parameters in the rational part such as:

$$\int \frac{x/(x-1)}{x+L_n(x)}\, dx.$$

However, such integrals rarely arise in quantum electrodynamics although they may occur in other fields. What happens more frequently is that the partial fraction denominators cannot be factored into linear factors and hence it is necessary to introduce Spence functions with irrational arguments or a new class of functions to complete the analytic integration over one variable. A further analytic integration of such expressions would involve different methods from those we have considered here. In extending our ideas to sixth order QED calculations, for instance, it is not clear at present whether answers will come out only in terms of Spence functions. However, all reported analytic evaluation of contributions to the sixth order anomalous magnetic moment have so far been of this form [11, 12]. In particular, the recent work of Levine and Roskies [12] bears out this conjecture. Their method uses an inspired technique for transforming the integrals, but relies eventually on a partial fraction expansion similar to the one we have discussed in this paper for getting the required answer. It would of course be tempting to guess that the whole sixth order magnetic moment is expressible only in terms of Spence functions, but this remains to be proven.

```
%APPENDIX A

                    %The RSIN Program

%Some basic declarations and definitions;

OFF MCD;

OPERATOR F,LI,LN,LN1,INT,PSI1,SL,SL1,SL2,ZETA;

LINEAR INT;

LET LI(2,1)= PI**2/6,
    LI(3,1)= ZETA 3;

FOR ALL X LET DF(LN1 X,X)=-1/(1-X);    %LN1 X= LN(1-X);

FOR ALL N,X LET LN(X**N)=N*LN(X);

PROCEDURE PSI(M,N,X);
  FOR I:=1:N-1 SUM X**I/I**M;

FOR ALL M LET SL(M,1)= PSI(1,M,1);

PROCEDURE ODDSUM N;
   FOR I:= 0:N SUM 1/(2*I+1);

%Definition of factorials;

ARRAY FAC(20);
FAC(0):=1$
FOR I:=1:20 DO FAC(I):=I*FAC(I-1);

PROCEDURE BFAC(M,N);
   FAC M/(FAC N*FAC(M-N));

%Generating functions for X and Y integrals;

PROCEDURE GEN(A,B,M,N);
   DF(DF((A-B)**(-1)*(LN1 B-LN1 A),A,M-1),B,N-1)
        /(FAC(M-1)*FAC(N-1));

PROCEDURE IGEN(L,M,N);
   DF(DF(DF((LN1(C1)-LN1(D))/((C1-C2)*(D-C1))
              -(LN1(C2)-LN1(D))/((C1-C2)*(D-C2)),
            D,L-1),C1,M-1),C2,N-1)
       /(FAC(L-1)*FAC(M-1)*FAC(N-1));

%Rules for integration over X;

FOR ALL L,M,N,K LET

   INT(X*(1-D*X)**(-1)*(1-C1*X)**(-1)*(1-C2*X)**(-1),X)=
        IGEN(1,1,1),

   INT(X**K*(1-D*X)**L*(1-C1*X)**M*(1-C2*X)**N,X)=
        IF K=-(L+M+N+2) THEN IGEN(-L,-M,-N)
         ELSE INT1(X,K,L,M,N);

FOR ALL A,B,C,L,M,N,K LET

   INT((1-A*X)**(-1)*(1-B*X)**(-1),X)=GEN(A,B,1,1),

   INT(X*(1-A*X)**(-2)*(1-B*X)**(-1),X)=GEN(A,B,2,1),

   INT(X**L*(1-A*X)**M*(1-B*X)**N,X)=
        IF L=-(M+N+2) THEN GEN(A,B,-M,-N)
         ELSE INT2(X,A,B,L,M,N),

   INT((1-D*X)**(-2)*LN(F(X)),X)=SL1(1),

   INT(X*(1-D*X)**(-3)*LN(F(X)),X)=SL1(2)/2,

   INT(X**N*(1-D*X)**M*LN(F(X)),X)=
        IF N=-M-2 THEN -SL1(-M-1)/(M+1) ELSE INT3(X,M,N);


%Rules for integration over W and Z;

FOR ALL M,N,X,Y LET

   INT(1,X)= 1,

   INT(X,X)= 1/2,

   INT(X**N,X)= IF N<0 THEN 0 ELSE 1/(N+1),

   INT((1-X)**N,X)= IF N<0 THEN 0 ELSE 1/(N+1),

   INT((1-Y*X)**N,X)= IF Y=2 THEN 0
                       ELSE IF N=-1 THEN -SL(1,Y)
                       ELSE (1-(1-Y)**(N+1))/(Y*(N+1)),

   INT(LN(1-X),X)= -1,

   INT((1-X)**N*LN(1-X),X)= IF N<0 THEN 0 ELSE -1/(N+1)**2,

   INT((1-2*X)**N*LN(1-X),X)=
        IF N=-1 THEN 3*LI(2,1)/4
         ELSE IF REMAINDER(N,2)=0 THEN 0
         ELSE -ODDSUM((-N-3)/2)/(N+1),
```

```
INT(SL(M,Y*X),X)=
   IF M=1 THEN -LI(2,Y)/Y ELSE -(SL(M,Y)-SL(1,Y))/(M-1),

INT(X*SL(M,Y*X),X)= -(SL(M,Y)-SL(2,Y))/(M-2),

INT(X**N*SL(M,Y*X),X)= IF N=M-1 THEN -LI(2,Y)+PSI(2,N,Y)
                        ELSE -(SL(M,Y)-SL(N+1,Y))/(M-N-1),

INT(LI(2,X),X)= LI(2,1)-1,

INT(X*LI(2,X),X)=LI(2,1)/2-3/8,

INT(X**N*LI(2,X),X)=
   IF N=-1 THEN LI(3,1)
    ELSE IF N<0 THEN (LI(2,1)-PSI(2,-N,1)+SL(-N,1)/(N+1))/(N+1)
    ELSE (LI(2,1)/(N+1)-SL(N+2,1))/(N+1),

INT((1-X)**N*LI(2,X),X)=
   IF N=-1 THEN -2*LI(3,1) ELSE PSI(2,-N,1)/(N+1),

INT(LI(2,1-(1-X)**(-1)),X)= -LI(2,1),

INT(X*LI(2,1-(1-X)**(-1)),X)= -LI(2,1)-1/2,

INT(X**N*LI(2,1-(1-X)**(-1)),X)=
     IF N=-1 THEN -2*LI(3,1)
      ELSE IF N=-2 THEN -2
      ELSE IF N=-3 THEN -7/4
      ELSE (PSI(2,-N,1)-SL(-N,1)/(N+1)
           + (FOR I:=2:(-N-1) SUM SL(I,1)/(I-1))/(N+1))/(N+1),

INT(LI(2,2-X**(-1)),X)= -LI(2,1)/2,

INT(X*LI(2,2-X**(-1)),X)=LI(2,1)/8,

INT((1-X)**N*LI(2,2-X**(-1)),X)=
   IF N=-1 THEN -3/4*LI(3,1)-3*LI(2,1)*LN 2
    ELSE 2*LI(2,1)*(2**(-N-1)-1)/(N+1)
          + (FOR K:=0:(-N-2) SUM BFAC(-N-2,K)/(N+K+1)**3)
          - FOR K:=1:(-N-2) SUM (2**K-1)*SL(-N-K,1)
                                 /(K*(N+K+1)),

INT((1-2*X)**N*LI(2,2-X**(-1)),X)=
   IF N=-1 THEN -7/8*LI(3,1)-3/2*LI(2,1)*LN 2
    ELSE -(FOR K:=0:(-N-3) SUM BFAC(-N-2,K)*(-2)**(-N-2-K)
              *SL(-N-K-1,1)/(N+K+1)**2)
         + IF REMAINDER(N,2)=0
              THEN LI(2,1)/(N+1)
                   + FOR K:=0:((-N-4)/2) SUM
                       ODDSUM(K)/((K+1)*(2*K+N-3))
             ELSE  2*FOR K:=0:((-N-3)/2) SUM
                       ODDSUM(K)/((2*K+1)*(2*K+N+2));
```

```
PROCEDURE RSIN(FN,K,M,N);

 BEGIN SCALAR ANS;

  %The following abbreviations are used in this procedure:
  %      W1= 1-W, R1= -Z/Z1, R2= W1, R3= 1-W*Z, R4= -Z/Z2,
  %      Z0= 1, Z1= Z/(1-Z), Z2= (1-2*Z)/(-Z);

  %Integration over Y;

  ANS:=GEN(C,BT,N,K);

  %Integration over X;

  FOR ALL L LET(1-BT)**L=(W**2*X)**(-L)*((1-C1*X)*(1-C2*X))**L,
            (C-BT)**L=DX**(2*L)/A**L,
            LN1 BT= LN(F(X))+LN1(C);

  ANS:=X**(2*K+N-M-3)
       *SUB(A=W**2*X,C=CB*X,DX=(1-D*X),DX**M*ANS/A**K);

  FOR ALL L CLEAR (1-BT)**L,(C-BT)**L,LN1 BT;

  ANS:=FN*INT(ANS,X);

  %Integration over W;

  FOR ALL L LET
      (1-D)**L=(W*W1*Z)**L, (D-CB)**L=(W**2*Z*R1)**L,
      (1-CB)**L=(W*Z*R3)**L, (1-C1)**L=(W*Z)**L,
      (1-C2)**L=(W*Z*(1-W*Z2))**L, (C1-C2)**L=(W**2*R4)**L,
      (D-C1)**L=(W**2*Z)**L, (D-C2)**L=(W**2*R1)**L,
      (CB-D)**L=(-W**2*Z*R1)**L,
     SL1(L)=(W*W1*Z)**(-L)*(PAR+SL2(L));

  ANS := ANS;

  FOR ALL L CLEAR (1-D)**L,(D-CB)**L,(1-CB)**L,
  (1-C1)**L,(1-C2)**L,(C1-C2)**L,(D-C1)**L,(D-C2)**L,
  (CB-D)**L,SL1(L);

  ANS1 := SUB(PAR=0,ANS);

  ANS2 := SUB(W=W1,W1=W,R2=W,R3=1-W1*Z,ANS-ANS1)/PAR;

  LET LN1(C1)=LN(W*Z),
      LN1(C2)=LN(W*Z)+LN1(W*Z2),
      LN1(CB)=LN(W*Z)+LN1(W*Z);
      LN1(D)=LN(W*Z)+LN1(W*Z0);
```

```
FOR ALL L LET
   SL2(L)= -(W1/W)**L*(Z1**L*LN1(W*Z2)-(Z/Z1)**(-L)*LN1(W*Z)
                              -Z1*PSI1(L)*LN1(W*Z0)
          + Z1*FOR J:= 1:L-2 SUM (W/W1)**J*PSI1(L-J)/J),

   PSI1(L)= (1-1/R1)**(L-1)+(-1)**L*(1+PSI(0,L,1/R1));


ANS:= SUB(W1=1-W,ANS1-(LN1(W*Z1)+LN1(W*Z0))*ANS2
             +LN1(W*Z2)*SUB(Z=R1,Z1=1/Z1,Z2=Z1*Z2,ANS2));

CLEAR ANS1,ANS2;

CLEAR LN1 C1,LN1 C2, LN1 CB,LN1 D;

FOR ALL L CLEAR SL2(L);

ANS := SUB(R1=-Z/Z1,R2=W1,R3=1-W*Z,R4=-Z*Z2,ANS);

%Partial fraction expansion for W;

LET W*(1-W)**(-1)=(1-W)**(-1)-1,
    W**(-1)*(1-W)**(-1)=W**(-1)+(1-W)**(-1),
     (1-W)**(-1)*(1-W*Z2)**(-1)=-Z1*((1-W)**(-1)-Z2/(1-W*Z2));

FOR ALL X LET W**(-1)*(1-W*X)**(-1)=W**(-1)+X*(1-W*X)**(-1),
              W*(1-W*X)**(-1)=X**(-1)*((1-W*X)**(-1)-1);

ANS:=ANS;

OFF MATCH;

%Generation of SL's in W;

FOR ALL L,X LET W**L*LN1(X*W)= IF L>0 THEN W**(L+1)*X*SL(1,W*X)
               ELSE X**(-L)*SL(-L,W*X)-W**L*PSI(1,-L,W*X);

ANS:=ANS;

FOR ALL L CLEAR PSI1(L);

OFF MATCH;

FOR ALL X LET LN1(W*X)=W*X*SL(1,W*X);

ANS := SUB(Z0=1,INT(ANS,W));

FOR ALL X CLEAR LN1(W*X);

%Integration over Z;

FOR ALL L,X LET SL(L,X)=X**(-L)*(LN(1-X)+PSI(1,L,X));

ANS := SUB(Z1=-Z/(1-Z),Z2=(1-2*Z)/(-Z),ANS);

FOR ALL L,X CLEAR SL(L,X);

%Partial fraction expansion for Z;

LET Z*(1-Z)**(-1)=(1-Z)**(-1)-1,
    Z**(-1)*(1-Z)**(-1)=Z**(-1)+(1-Z)**(-1),
    Z*(1-2*Z)**(-1)=((1-2*Z)**(-1)-1)/2,
    Z**(-1)*(1-2*Z)**(-1)=2*(1-2*Z)**(-1)+Z**(-1),
    (1-Z)**(-1)*(1-2*Z)**(-1)=2*(1-2*Z)**(-1)-(1-Z)**(-1);

ANS := ANS;

OFF MATCH;

ANS1 := SUB(LN(1/Z-1)=0,ANS);

ANS2 := (ANS-ANS1)/LN(1/Z-1);

ANS  := ANS1+(ANS2-SUB(Z=1-Z,ANS2))*LN(1-Z);

FOR ALL L LET Z**L*LN(1-Z)=IF L>0 THEN Z**(L+1)*SL(1,Z0*Z)
                          ELSE SL(-L,Z0*Z)-Z**L*PSI(1,-L,Z),
              (2*Z-1)**L= (-1)**L*(1-2*Z)**L;

ANS := ANS;

OFF MATCH;

RETURN SUB(Z0=1,INT(ANS,Z));

END;

%Examples;

RSIN(Z*W**2,1,0,1); %Value is PI**2/12;

RSIN(Z*W**3,1,1,2); %Value is -3/8*PI**2-15/8*ZETA 3+3/4*PI**2*LN 2+1;

RSIN(Z*W**3,1,1,3); %Value is -5/72*PI**2+11/12;

RSIN(Z*W**2,1,1,3); %Value is 1/2;

RSIN(Z**2*W**3,1,0,2); %Value is 1/6*PI**2-1/3*PI**2*LN 2+1/2*ZETA 3+1/3;

END;
```

```
%APPENDIX B

%                              ZINT
%                     A Program for Finding
%      the Hadamard Finite Part of some Divergent Integrals;


%Some basic declarations and definitions;

OFF MCD;

OPERATOR INT,LI,LN,SL,ZETA;

LINEAR INT;

LET LI(2,1)= PI**2/6,
    LI(3,1)= ZETA 3;

PROCEDURE PSI(M,N,X);
  FOR I:= 1:N SUM X**I/I**M;

FOR ALL M LET SL(M,1)= PSI(1,M,1);

PROCEDURE ODDSUM N;
   FOR I:= 0:N SUM 1/(2*I+1);

%Definition of factorials;

ARRAY FAC(20);
FAC(0):=1$
FOR I:=1:20 DO FAC(I):=I*FAC(I-1);

PROCEDURE BFAC(M,N);
   FAC M/(FAC N*FAC(M-N));

%Integration rules;

FOR ALL N,X,Y LET

    INT(LI(2,X),X)= LI(2,1)-1,

    INT(X*LI(2,X),X)=LI(2,1)/2-3/8,

    INT(X**N*LI(2,X),X)=
       IF N=-1 THEN LI(3,1)
        ELSE IF N<0 THEN (LI(2,1)-PSI(2,-N,1)+SL(-N,1)/(N+1))/(N+1)
        ELSE (LI(2,1)/(N+1)-SL(N+2,1))/(N+1),

    INT((1-X)**N*LI(2,X),X)=
       IF N=-1 THEN -2*LI(3,1) ELSE PSI(2,-N,1)/(N+1),

    INT(LI(2,1-(1-X)**(-1)),X)= -LI(2,1),
```



```
INT(X*LI(2,1-(1-X)**(-1)),X)= -LI(2,1)-1/2,

INT(X**N*LI(2,1-(1-X)**(-1)),X)=
     IF N=-1 THEN -2*LI(3,1)
      ELSE IF N=-2 THEN -2
      ELSE IF N=-3 THEN -7/4
      ELSE (PSI(2,-N,1)-SL(-N,1)/(N+1)
              + (FOR I:=2:(-N-1) SUM SL(I,1)/(I-1))/(N+1))/(N+1),

INT(LI(2,2-X**(-1)),X)= -LI(2,1)/2,

INT(X*LI(2,2-X**(-1)),X)=LI(2,1)/8,

INT((1-X)**N*LI(2,2-X**(-1)),X)=
   IF N=-1 THEN -3/4*LI(3,1)-3*LI(2,1)*LN 2
    ELSE 2*LI(2,1)*(2**(-N-1)-1)/(N+1)
          + (FOR K:=0:(-N-2) SUM BFAC(-N-2,K)/(N+K+1)**3)
          - FOR K:=1:(-N-2) SUM (2**K-1)*SL(-N-K,1)
                                  /(K*(N+K+1)),

INT((1-2*X)**N*LI(2,2-X**(-1)),X)=
   IF N=-1 THEN -7/8*LI(3,1)-3/2*LI(2,1)*LN 2
    ELSE -(FOR K:=0:(-N-3) SUM BFAC(-N-2,K)*(-2)**(-N-2-K)
             *SL(-N-K-1,1)/(N+K+1)**2)
        + IF REMAINDER(N,2)=0
            THEN LI(2,1)/(N+1)
                 + FOR K:=0:((-N-4)/2) SUM
                    ODDSUM(K)/((K+1)*(2*K+N-3))
           ELSE  2*FOR K:=0:((-N-3)/2) SUM
                    ODDSUM(K)/((2*K+1)*(2*K+N+2));


%Examples;

INT(LI(2,X)*X**(-3),X); %Value is -PI**2/12+41/36;

INT(LI(2,X)*(1-X)**(-2),X); %Value is -5/4;

INT(LI(2,(2-X**(-1)))*(1-X)**(-4),X);
      %Value is -7*PI**2/9+203/108;

INT(LI(2,(2-X**(-1)))*(1-2*X)**(-1),X);
      %Value is -PI**2*LN(2)/4-7*ZETA(3)/8;


END;
```

Acknowledgment

The authors would like to thank Dr. K. Kauffmann for some very useful discussions.

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Printed in Belgium